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Lie Matrix Groups: The Flip Transpose Group

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Lie Matrix Groups: The Flip Transpose Group

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Abstract. In this paper we explore the geometry of persymmetric matrices. These matrices are defined in a similar way to orthogonal matrices except the (flip) transpose is taken over the skew diagonal. We give a proof that persymmetric matrices form a Lie group.

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1 Introduction

Lie groups, named after Sophus Lie, span two fields of mathematics as they are groups that are also differentiable manifolds. The crossover between smooth surfaces and group operations makes Lie group theory a complex, yet very useful, mathematical theory. For example, Lie groups provide a framework for analyzing the continuous symmetries of differential equations. Many different mathematical objects can be Lie groups, but for the purposes of this paper we will be focusing on Lie matrix groups.

An example of a Lie matrix group is the orthogonal group. The orthogonal group is the set of all $n \times n$ matrices with non-zero determinant that have the property $A^T A = I$. The orthogonal group is classified as a Lie matrix group because it satisfies the group axioms and is an n-dimensional manifold. Geometrically, the orthogonal group represents rotations in the plane.

This paper works up to the definition of a Lie group by understanding group theory and investigating Lie algebras. Along the way, we will explore the effects of the flip transpose on matrices, and will introduce a new group of matrices, the flip transpose group. (The flip transpose is formed by reflecting along the skew-diagonal.)

In Section 2 we provide key definitions and examples, and we define the matrix groups that we will later explore. This section also contains an example of a Lie group that will aid in the understanding of the crossover between manifolds and group theory. Section 3 contains proofs showing that certain sets of matrices are groups. In Section 4 we introduce a new matrix group, the flip transpose group, and we provide proofs of properties pertaining to this group. Section 5 introduces a Lie algebra and includes necessary terms, examples and proofs all relating the matrix groups we are working with. In Section 6 we investigate what a Lie group is, and in Section 7 we prove that the flip transpose group is a Lie group. Lastly, in Section 8 we provide further considerations for research in this area of mathematics.

2 Background

The following terms and examples are preliminary to the understanding of this research; they will lead up to the concept at the center of the research: the Lie matrix group. There are three main topics of mathematics that will be covered throughout this paper: matrices, groups, and surfaces.

2.1 Basic Definitions and Examples

The notation A_{ij} will be used often throughout this paper and represents the element in the i-th row and j-th column of a given matrix A. The notation a_{ij} will also be used to represent the element in the i-th row and j-th column, when convenient.

Definition 1 (Standard Diagonal)**.** The *standard diagonal of an $n \times n$ matrix* is the set of entries of the form a_{ii} where $1 \leq i \leq n$. The entries on the standard diagonal go from the top left corner of a matrix to the bottom right corner.

Definition 2 (Transpose)**.** The *transpose of a matrix A with entries a_{ij}* is the matrix A^T whose entries are a_{ji}.

Example 1.

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

The following properties of transposes are also important to note, as they are useful when proving theorems and will be utilized often throughout the paper.

Property 1. $(AB)^T = B^T A^T$

Property 2. $(A^{-1})^T = (A^T)^{-1}$

Property 3. $(A^T)^T = A$

Definition 3 (Symmetric Matrices)**.** A matrix is *symmetric* if it equals its transpose, $A = A^T$.

Definition 4 (Skew-Symmetric Matrices)**.** A matrix is *skew-symmetric* if it equals its negative transpose, $A = -A^T$.

The transpose is an operation on matrices that comes up often in linear algebra. For the purposes of this paper, we will look at a different type of transposition, the flip transpose, and the different kinds of symmetries that can exist with this new transposition. The motivation for looking at the flip transpose comes from a 1997 paper of Reid [5].

Definition 5 (Skew Diagonal)**.** The *skew diagonal of an $n \times n$ matrix* is the set of entries going from the top right corner of the matrix to the bottom left corner.

Definition 6 (Flip Transpose)**.** *[1]* The *flip transpose of a matrix A with entries a_{ij}* is the matrix A^F whose entries are $a_{n+1-j,n+1-i}$. The flip transpose of a matrix is formed by reflecting along the skew-diagonal.

Example 2.

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ then } A^F = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}.$$

Definition 7 (Persymmetric)**.** *[5]* A matrix is *persymmetric* if it equals its flip transpose, $A = A^F$.

Definition 8 (Per-antisymmetric)**.** *[5]* A matrix is *per-antisymmetric* if it equals its negative flip transpose, $A = -A^F$.

The next important part of the project will focus on group structure, specifically matrix group structure. Therefore, we need to first understand what a group is and then look at some specific matrix groups.

Definition 9 (Group). *[3]* A *group* is a set, G, together with an operation, $\cdot$, that combines any two elements a and b to form another element. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms.

- Closure:

  For all a, b in G, the result of the operation, $a \cdot b$, is also in G.

- Associativity:

  For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

- Identity Element:

  There exists an element e in G, such that for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds.

- Inverse Element:

  For each a in G, there exists an element b in G such that $a \cdot b = b \cdot a = e$.

2.2 Matrix Groups

There exists different matrix groups that coincide with Lie theory, but for the purposes of this paper we will be investigating the general linear group and orthogonal group in depth. These two groups provide a template for working with the flip transpose that we are interested in.

Definition 10 (General Linear Group). *[2]* The *general linear group of degree n,* denoted $GL_n(\mathbb{R})$, is the set of all $n \times n$ matrices such that the determinant of the matrix is not zero.

Definition 11 (Orthogonal Group). *[2]* The *orthogonal group of degree n,* denoted $O_n(\mathbb{R})$, is the set of all $n \times n$ matrices in $GL_n(\mathbb{R})$ such that $A^T A = I$.

We can see that $O_n(\mathbb{R})$ is a subset of $GL_n(\mathbb{R})$ since if $A^T A = I$, then A has an inverse. Hence $\det A \neq 0$.

We can now prove that these sets of matrices are in fact groups and begin to look at the new matrix group involving the flip transpose.

3 Proofs for Matrix Groups

Since $O_n(\mathbb{R})$ is contained in $GL_n(\mathbb{R})$, and the general linear group is a group with respect to multiplication, showing that the orthogonal group is a subgroup of the general linear group will show that it is a group. We want an understanding of the relationship between the orthogonal group and the general linear group because the group involving the flip transpose will work similarly. First, we must understand what a subgroup is.

Definition 12 (Subgroup)**.** *[3]* Let G be a group with respect to the binary operation $\cdot$. A subset H of G is called a *subgroup of G* if H forms a group with respect to the binary operation $\cdot$ that is defined in G.

The following theorem will provide the conditions that need to be met to be a subgroup.

Theorem 1. *A subset H of the group G is a subgroup if and only if these conditions are satisfied:*

(a) *H is nonempty.*

(b) $x \in H$ *and* $y \in H$ *imply* $xy \in H$.

(c) $x \in H$ *implies* $x^{-1} \in H$.

Theorem 2. *The orthogonal group is a group.*

Proof.

(a) This set is nonempty because the identity matrix, I_n, satisfies the properties of this set.

(b) Given matrix A and matrix B in $O_n(\mathbb{R})$, AB must also be in the set. Thus we show that

$$(AB)^T(AB) = I.$$

Using Property 1 of the transpose we have

$$(AB)^T(AB) = (B^T A^T)(AB) = B^T(A^T A)B = B^T IB = B^T B = I$$

as desired.

(c) Given matrix $A \in O_n(\mathbb{R})$, since $A^T A = I$, we need to show that A^T is in $O_n(\mathbb{R})$. Using Property 3 of the transpose we have

$$A^T(A^T)^T = I$$

$$A^T(A) = I$$

as desired.

The orthogonal group is a subgroup of the general linear group and therefore is a group.

4 Flip Transpose Group

We now define a new group that will be the focus of this paper. This group is similar to the orthogonal group, but utilizes the flip transpose operation that was defined in Section 2.

Definition 13 (Flip Transpose Group)**.** The *flip transpose group of degree n,* denoted $F_n(\mathbb{R})$, is the set of all matrices in $GL_n(\mathbb{R})$ such that $A^F A = I$.

We need to prove two properties of the flip transpose to aid in the proof that $F_n(\mathbb{R})$ is a group. Specifically, we'll show that $(AB)^F = B^F A^F$ and $(A^F)^F = A$. Before doing so, however, we need an example to get an understanding of the notation in the flip transpose group.

Let's look at a 3×3 matrix A written with indexed entries,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then A^F would have the following form

$$\begin{bmatrix} a_{33} & a_{23} & a_{13} \\ a_{32} & a_{22} & a_{12} \\ a_{31} & a_{21} & a_{11} \end{bmatrix}.$$

The entry a_{11} moves to the spot a_{33} in A^F, a_{12} moves to a_{23}, and so on. Our conjecture is that the new placement takes on the form $A_{n+1-j,n+1-i}$ where n is the size of the matrix (3 in this case). If we apply this to a few different entries, we see that it works.

For the entry a_{11}: $a_{3+1-1,3+1-1} = a_{33}$.
For the entry a_{21}: $a_{3+1-1,3+1-2} = a_{32}$.
For the entry a_{32}: $a_{3+1-2,3+1-3} = a_{21}$.

Using the definition of flip transpose that was demonstrated with this example we can move on to proving the following theorem.

Theorem 3 (Flip Transpose Theorem)**.** *For $n \times n$ matrices A and B,*

(a) $(AB)^F = B^F A^F$

(b) $(A^F)^F = A$.

Proof.

(a) We know that $((AB)^F)_{ij} = (AB)_{(n+1-j,n+1-i)}$.

When expanded,

$$(AB)_{(n+1-j,n+1-i)} = a_{(n+1-j),1}b_{1,(n+1-i)} + a_{(n+1-j),2}b_{2,(n+1-i)} + \ldots + a_{(n+1-j),n}b_{n,(n+1-i)}$$

where a and b represent the individual entries in the matrices A and B respectively.

Let $a'_{ij} = a_{(n+1-j),(n+1-i)}$, to represent A^F and let $b'_{ij} = a_{(n+1-j),(n+1-i)}$, to represent B^F.

Therefore,

$$\begin{aligned}
(B^F A^F)_{ij} &= b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \ldots + b'_{in}a'_{nj} \\
&= b_{1,(n+1-i)}a_{(n+1-j),1} + b_{2,(n+1-i)}a_{(n+1-j),2} + \ldots + b_{n,(n+1-i)}a_{(n+1-j),n} \\
&= a_{(n+1-j),1}b_{1,(n+1-i)} + a_{(n+1-j),2}b_{2,(n+1-i)} + \ldots + a_{(n+1-j),n}b_{n,(n+1-i)} \\
&= ((AB)^F)_{ij}
\end{aligned}$$

as desired.

Thus, $(AB)^F = B^F A^F$.

Notice: The multiplication of entries in a matrix is commutative because we are multiplying real numbers. Hence, we can switch the placement of a and b so that the product matches the expansion of $((AB)^F)_{ij}$ exactly.

(b) We know that $(A^F)_{ij} = a_{(n+1-j),(n+1-i)}$.

Therefore,

$$\begin{aligned}
(A^F)^F &= (a_{(n+1-j),(n+1-i)})^F \\
&= a_{(n+1-(n+1-i)),(n+1-(n+1-j))} \\
&= a_{ij}
\end{aligned}$$

as desired.

Thus, $(A^F)^F = A$.

We now show that the set of all $n \times n$ persymmetric matrices of degree n forms a group.

Theorem 4. *The set $F_n(\mathbb{R})$ is a subgroup of the general linear group.*

Proof.
$F_n(\mathbb{R})$ is a subset of $GL_n(\mathbb{R})$ by definition. $F_n(\mathbb{R})$ contains the identity and therefore it is nonempty.

(1) Closure:

Let A and B be matrices in this set. Then their product, $(AB)^F(AB)$, must be shown to also be equal to the identity matrix, that is, $(AB)^F(AB) = I$.

Left Hand Side

$$\begin{aligned} &= (B^F A^F)(AB) \\ &= B^F(A^F A)B \\ &= B^F I B \\ &= B^F B \\ &= I \end{aligned}$$

as desired.

(2) Associativity:

Matrix multiplication is associative.

(3) Identity Element:

This set contains the identity matrix, I_n since $(I_n)^F I_n = I_n$.

Claim: $A^F = A^{-1}$, since $A^F A = I$, we know A^F is the left inverse.

Notice: $AA^F A = A$. If we multiply on the right side by A^{-1}, we get $AA^F = I$, showing A^F is also a right inverse and thus $A^F = A^{-1}$.

(4) Inverse Element: Given matrix $A \in F_n(\mathbb{R})$, since $A^F A = I$, we need to show that $A^F \in F_n(\mathbb{R})$,

$$A^F(A^F)^F = I$$

$$A^F(A) = I$$

as desired.

Therefore, the set of all $n \times n$ matrices satisfying $A^F A = I$ is a group.

It is also important to notice that $F_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$. The flip transpose group satisfies the three conditions from Theorem 1 that must be met in order to be a subgroup and is therefore a subgroup of the general linear group. This result will be important later when we investigate Lie groups.

5 Lie algebras

Before diving into Lie groups, we first need an understanding of what a Lie algebra is and how it works with the orthogonal group and the flip transpose group.

Definition 14 (Lie algebra)**.** *[2]* A $\mathbb{R}$-*Lie algebra* consists of a vector space $\mathfrak{a}$ equipped with a $\mathbb{R}$-bilinear map $[,] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ such that for $x, y, z \in \mathfrak{a}$,

$$[x, y] = -[y, x], \text{(Skew Symmetry)}$$
$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0 \text{(Jacobi Identity)}.$$

To understand the Lie algebra definition, we must understand what a bilinear map is.

Definition 15 (Bilinear Map)**.** *[2]* A *bilinear map* is a function combining elements of two vector spaces to yield an element of a third vector space that is linear in each of its arguments.

Matrix multiplication is an example of a bilinear map because $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$. Matrix multiplication is linear in its arguments for the following reasons

$$M(A, B) = AB$$

$$M(A + D, B) = M(A, B) + M(D, B)$$
$$M(A, B + D) = M(A, B) + M(A, D)$$
$$M(cA, B) = cM(A, B)$$
$$M(A, cB) = cM(A, B)$$

where M is the bilinear map, A, B and D are matrices contained in $M_n(\mathbb{R})$, and c is a scalar. The bilinear map we use is the commutator of two matrices. Given two matrices $A, B \in M_n(\mathbb{R})$, their commutator is

$$[A, B] = AB - BA.$$

Before we explore more about the Lie algebras for certain matrix groups, we must understand what a tangent space is.

Definition 16 (Tangent Space)**.** *[2]* The *tangent space to G at $U \in G$* is

$$T_U G = \{\gamma'(0) \in M_n(\mathbb{R}) : \gamma \text{ a differentiable curve in } G \text{ with } \gamma(0) = U\}$$

where G is a matrix group.

The concepts of a Lie algebra and tangent space go hand in hand. In fact, the notation $\mathfrak{g}$ symbolizes both the tangent space and the real Lie algebra. We will show that

$$\mathfrak{g} = T_I G,$$

the tangent space to a group of matrices at the identity is a Lie algebra. Since we know a Lie algebra is a vector space equipped with a bilinear map, we know that the tangent space is also a real vector space of $M_n(\mathbb{R})$. For each matrix group G, there is a Lie algebra $\mathfrak{g} = T_I G$. We look at the Lie algebra of $O_n(\mathbb{R})$ because the proof is similar to the proof for $F_n(\mathbb{R})$. Before doing so, we must prove the following theorem, which demonstrates that the tangent space, $\mathfrak{g}$, is a Lie algebra of a matrix group. The theorem and proof are from a 2000 paper of Baker [2].

Theorem 5. *If $G \leq GL_n(\mathbb{R})$ is a matrix subgroup, then $\mathfrak{g}$ is a $\mathbb{R}-$Lie subalgebra of $M_n(\mathbb{R})$.*

The goal in the following proof is to find what kind of matrices exist in the tangent space, $\mathfrak{g}$, of $O_n(\mathbb{R})$. We explore $O_n(\mathbb{R})$ because the property of this group is similar to the property of $F_n(\mathbb{R})$.

Before doing so, we need an understanding of a definition and proposition that will be used in the following proof. Both are from a 2000 paper of Baker [2].

Definition 17. The *power series,* $\text{Exp}(X)$, is defined as

$$\text{Exp}(X) = \sum_{n \geq 0} \frac{1}{n!} X^n.$$

Proposition 1. *For an $n \times n$ matrix A and any real number t,*

(a) $(\text{Exp}(tA))^T = \text{Exp}(tA^T)$

(b) $(\text{Exp}(tA))^F = \text{Exp}(tA^F)$

(c) $\text{Exp}(A) \in GL_n(\mathbb{R})$ *and* $(\text{Exp}(A))^{-1} = \text{Exp}(-A)$.

Proof. The proof of (a) follows similarly to the proof of (b) (which is below). The proof of (c) can be found in a 2000 paper of Baker [2].

Proof of (b):

$$\begin{aligned}
(\text{Exp}(tA))^F &= (I + tA + \frac{(tA)^2}{2!} + ...)^F \\
&= I + (tA)^F + \frac{(t^2A^2)^F}{2!} + ... \\
&= I + (tA)^F + \frac{((tA)^F)^2}{2!} + ... \\
&= \text{Exp}(tA^F).
\end{aligned}$$

Theorem 6. *The tangent space of $O_n(\mathbb{R})$ is made up of skew-symmetric matrices.*

Before proving this theorem, it is important to remember the definition of skew-symmetric matrices that was provided in Section 2. A skew-symmetric matrix is a matrix that is equal to its negative transpose, $A = -A^T$. The group of skew-symmetric matrices is notated as $Sk - Sym_n(\mathbb{R})$.

Proof. Given a curve $\alpha : (a, b) \to O_n(\mathbb{R})$ satisfying $\alpha(0) = I$ where (a, b) is an interval of real numbers, we have,

$$\frac{d}{dt} \alpha(t)^T \alpha(t) = 0.$$

We are taking the derivative of the curve that satisfies the property of the orthogonal group. It is important to notice that $\alpha(t)$ is a matrix. Using the Product Rule, we have

$$\alpha'(t)^T\alpha(t) + \alpha(t)^T\alpha'(t) = 0,$$

which can be simplified to

$$\alpha'(0)^T + \alpha'(0) = 0$$

since it was given that the curve has the property $\alpha(0) = I$. Thus we have $\alpha'(0)^T = -\alpha'(0)$, that is, $\alpha'(0)$ is skew-symmetric. Thus,

$$\mathfrak{o}_n(\mathbb{R}) = T_I O_n(\mathbb{R}) \subseteq Sk - Sym_n(\mathbb{R}).$$

The Lie algebra of the orthogonal group is the tangent space to $O_n(\mathbb{R})$ which is contained in the set of all $n \times n$ real skew-symmetric matrices. On the other hand, say we are given a matrix $A \in Sk - Sym_n(\mathbb{R})$ and we consider the curve

$$\alpha : (-\epsilon, \epsilon) \to GL_n(\mathbb{R})$$

such that

$$\alpha(t) = \mathrm{Exp}(tA).$$

If we apply the property of the orthogonal group we have,

$$\begin{aligned}
\alpha(t)^T\alpha(t) &= (\mathrm{Exp}(tA))^T\mathrm{Exp}(tA) \\
&= \mathrm{Exp}(tA^T)\mathrm{Exp}(tA) \\
&= \mathrm{Exp}(-tA)\mathrm{Exp}(tA) \\
&= I.
\end{aligned}$$

This shows that $\alpha(t) \in O_n(\mathbb{R})$. Therefore, we can view this curve as

$$\alpha : (-\epsilon, \epsilon) \to O_n(\mathbb{R})$$

since it satisfies the conditions of the orthogonal group. Since $\alpha'(0) = A$, we know that A is in the tangent space and is a skew-symmetric matrix that is in $O_n(\mathbb{R})$. We can now say that

$$Sk - Sym_n(\mathbb{R}) \subseteq \mathfrak{o}_n(\mathbb{R}) = T_I O_n(\mathbb{R}),$$

so

$$\mathfrak{o}_n(\mathbb{R}) = T_I O_n(\mathbb{R}) = Sk - Sym_n(\mathbb{R}).$$

The Lie-algebra of $F_n(\mathbb{R})$ works similar to $O_n(\mathbb{R})$.

Theorem 7. *The tangent space of $F_n(\mathbb{R})$ is made up of per-antisymmetric matrices.*

Remember that per-antisymmetric matrices are matrices that are equal to their negative flip transpose, $A = -A^F$. The notation to represent the group of per-antisymmetric matrices is $Per - ASym_n(\mathbb{R})$. The Lie algebra for the $F_n(\mathbb{R})$ is notated as $\mathfrak{f}_n(\mathbb{R})$.

Proof. Given a curve $\alpha : (a, b) \to F_n(\mathbb{R})$ satisfying $\alpha(0) = I$ we have,

$$\frac{d}{dt}\alpha(t)^F\alpha(t) = 0,$$

and so

$$\alpha'(t)^F\alpha(t) + \alpha(t)^F\alpha'(t) = 0,$$

implying

$$\alpha'(0)^F + \alpha'(0) = 0$$

since

$$\alpha(0) = I \text{ and } \alpha(0)^F = I.$$

Thus we have $\alpha'(0)^F = -\alpha'(0)$, that is, $\alpha'(0)$ is per-antisymmetric.

Therefore,

$$\mathfrak{f}_n(\mathbb{R}) = T_IF_n(\mathbb{R}) \in Per - ASym_n(\mathbb{R}).$$

We can say that the matrices that make up the tangent space to $F_n(\mathbb{R})$ are contained in the set of all $n \times n$ per-antisymmetric matrices. To show that the tangent space is equal to the set of $n \times n$ per-antisymmetric matrices, we will exponentiate the curve. Consider the curve

$$\alpha : (-\epsilon, \epsilon) \to GL_n(\mathbb{R})$$

such that

$$\alpha(t) = \text{Exp}(tA).$$

If we apply the property of the flip transpose group we have,

$$\begin{aligned} \alpha(t)^F\alpha(t) &= (\text{Exp}(tA))^F\text{Exp}(tA) \\ &= \text{Exp}(tA^F)\text{Exp}(tA) \\ &= \text{Exp}(-tA)\text{Exp}(tA) \\ &= I. \end{aligned}$$

Therefore, we can view this curve as,

$$\alpha : (-\epsilon, \epsilon) \to F_n(\mathbb{R})$$

and can conclude that

$$Per - ASym_n(\mathbb{R}) \subseteq \mathfrak{f}_n(\mathbb{R}) = T_IF_n(\mathbb{R}).$$

We can now say that the tangent space of $F_n(\mathbb{R})$ consists of per-antisymmetric matrices, or

$$Per - ASym_n(\mathbb{R}) = T_IF_n(\mathbb{R}) = \mathfrak{f}_n(\mathbb{R}).$$

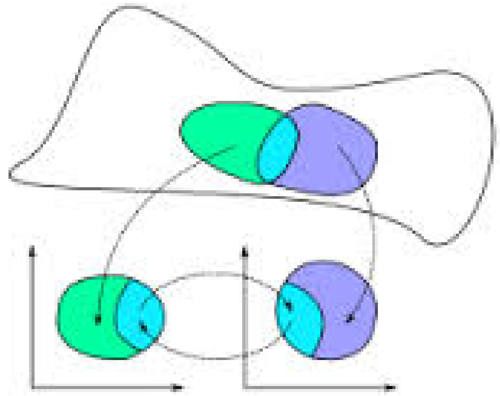

Figure 1: Smooth Maps

6 Lie Groups

Before we state the definition of a Lie group, we need an understanding of some background terms that are central to the concept of a Lie group.

Definition 18 (Smooth)**.** *[2]* A continuous map $g : V_1 \to V_2$ where each $V_k \subseteq \mathbb{R}^{m_k}$ is open, is called *smooth* if it is infinitely differentiable.

Example 3. Polynomials, sine, cosine and exponential functions are all smooth maps from $\mathbb{R}^1 \to \mathbb{R}^1$ because for any function in those classes there is an infinite number of derivatives that can be taken.

Definition 19 (Chart)**.** *[2]* A homeomorphism $f : U \to V$ where $U \subseteq M$, M is a set that will later be called a manifold, and $V \subseteq \mathbb{R}^n$ are open subsets, is called an *n-chart for U.*

In simpler terms, a chart is a function that takes us from an open set on a manifold to an open set in $\mathbb{R}^n$. A collection of charts is called an atlas. Charts and atlases are used often when working with manifolds because we need to take sets into spaces where we can perform operations, such as taking derivatives. Figure 1 demonstrates two different charts, each coming from open subsets of a manifold to $\mathbb{R}^2$.

Definition 20 (Manifold)**.** *[2]* A *smooth manifold of dimension* n is denoted by (M, U, F) where M is the set of points or space we are working with, U is the open covering in the space, and F is a chart or collection of charts.

For the purpose of this paper, the points in a given space we are working with are matrices and the charts are maps taking us from manifolds into $\mathbb{R}^n$.

Example 4. One example of a manifold of dimension 2 is the unit sphere in $\mathbb{R}^3$. The open sets are the 6 open hemispheres with a pole on an axis. The charts are the standard projection to the disk. i.e. if U is the upper open hemisphere then $f(u, v, \sqrt{1-u^2-v^2}) = (u, v)$.

Now we define a Lie group.

Definition 21 (Lie Group)**.** *[2]* Let G be a smooth manifold which is also a topological space with multiplication map mult: $G \times G \to G$ and inverse map inv: $G \to G$ and view $G \times G$ as the product manifold. Then G is a *Lie group* if mult, inv maps are smooth maps.

Our goal is to show that the flip transpose group is a Lie group. Since the flip transpose group is a subgroup of the general linear group, we need to first show that the general linear group is a Lie group. Once we show this, we use definitions and theorems that will guide us towards $F_n(\mathbb{R})$ as a Lie group. The ideas in the following proof are from a 2000 paper of Baker [2].

Theorem 8. *The general linear group is a Lie group.*

Proof. [2] We want to show that $GL_n(\mathbb{R})$ is a Lie group. First, $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$ is an open subset. This can be seen by noticing the determinant map, $\det : GL_n(\mathbb{R}) \to \mathbb{R}$, is a continuous function since it is a polynomial in the entries of the matrix. Also, notice that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}'\{0\})$. Since $\mathbb{R}'\{0\}$ is open and the determinant is continuous, we have $GL_n(\mathbb{R})$ is open. For charts we can take the open sets $U \subseteq GL_n(\mathbb{R})$ and the identity function Id: $U \to U$. The tangent space at each point $A \in GL_n(\mathbb{R})$ is just $M_n(\mathbb{R})$, so the notion of tangent space that we have previously discussed agrees here. The multiplication and inverse maps are smooth as they are defined by polynomial and rational functions between open subsets of $M_n(\mathbb{R})$. Therefore, $GL_n(\mathbb{R})$ is a Lie group.

The following definitions and theorem, both from a paper of Baker [2], will allow us to show that the orthogonal group and the flip transpose group are Lie groups.

Definition 22 (Lie Subgroup)**.** *[2]* Let G be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a *Lie subgroup of* G. It is then automatic that the restrictions to H of the multiplication and inverse maps on G are smooth, hence H is also a Lie group.

This definition is providing a shortcut to proving that subgroups of the general linear group are Lie groups. If we can show that the matrix groups are subgroups of $GL_n(\mathbb{R})$ (which was done in Section 3) and also show that they are submanifolds, then we know they are Lie subgroups. When shown to be a Lie subgroup, it automatically follows that it is a Lie group. So first, we define a submanifold.

Definition 23 (Submanifold)**.** *[2]* Let (M, U, F) be a manifold of dimension n. A subset $N \subseteq M$ is a *submanifold of dimension* k if for every $p \in N$ there is an open neighborhood $U \subseteq M$ of p and an n-chart $f : U \to V$ such that

$$p \in f^{-1}(V \cap \mathbb{R}^K) = N \cap U.$$

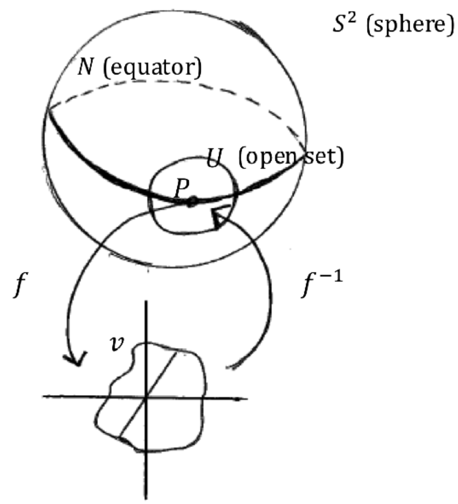

Figure 2: Submanifold

Example 5. The image in Figure 2 demonstrates how a submanifold works. In this example, the sphere is the manifold and the equator N is a subset of the sphere. We see that the darkened line segment of the equator N in the open set U around the point p represents $N \cap U$, which is also $f^{-1}(V \cap \mathbb{R}^K)$. Therefore, the equator is a submanifold of the sphere.

Now we know that we need to show that the orthogonal group and flip transpose group are submanifolds of the general linear group. The following theorem tells us how we can go about this.

Theorem 9 (Implicit Function Theorem for Manifolds)**.** *[2] Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimension n, n'. Suppose that for some $q \in M$, $dh_p : T_pM \to T_{h(p)}M'$ is surjective for every $p \in N = h^{-1}q$. Then $N \subseteq M$ is a submanifold of dimension $n - n'$ and the tangent space at $p \in N$ is given by $T_pN = \ker dh_p$.*

Therefore, for us to show that a matrix group is a submanifold, N, we need to show that the derivative of the map $F : GL_n(\mathbb{R}) \to \mathbb{R}^{\mathrm{n}}$ is surjective for every point p in the submanifold N, where $N = F^{-1}(q)$ for some q in $\mathbb{R}^{\mathrm{n}}$.

Let's first look at the orthogonal group. The ideas in the following proof come from a 2000 paper of Baker [2]. Before proving it, however, let's look at a 2×2 matrix that has the property $A^TA = I$. Recall that $O_n(\mathbb{R})$ is the solution set of a family of polynomial equations in n^2 variables arising from the matrix equation $A^TA = I$. For a 2×2 matrix we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ so } A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

So,

$$A^TA = I : \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Multiplying these matrices yields 4 expressions in 4 variables

$$F(A) = \begin{bmatrix} a^2+c^2-1 \\ ab+cd \\ ab+cd \\ b^2+d^2-1 \end{bmatrix}$$

where F is the map that is taking the matrix A as its input and giving a vector containing the resulting polynomial expressions as its output.

We notice that the second and third expressions are the same, so we can eliminate one of those expressions and will be left with 3 expressions. No matter the size of n, expressions will repeat. In fact, $\binom{n+1}{2}$ unique expressions will exist. Now we want to find the Jacobian for the map of $F(A) : GL_n(\mathbb{R})^n \to \mathbb{R}^n$, which is dF_A. To do so, we will use the following theorem and proposition as used in a 2000 paper of Baker [2].

Proposition 2 (Identity Check Trick)**.** *[2] Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup, M a smooth manifold and $F : GL_n(\mathbb{R}) \to M$ a smooth function with $F^{-1}q = G$ for some $q \in M$. Suppose that for every $B \in G, F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. If dF_I is surjective then dF_A is surjective for all $A \in G$ and $\ker dF_A = A\mathfrak{g}$.*

To use this proposition, we need to first prove that the hypothesis is true. In other words, we need to show that for every $B \in G, F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. The map we are working with, F, means we are applying the condition of the orthogonal group, which means $F(BC)$ and $F(C)$ are formed from the equations $(BC)^T(BC) = I$ and $C^TC = I$, respectively.

Proof. We want to show that $F(BC) = F(C)$, or $(BC)^T(BC) = I = C^TC$.

$$(BC)^T(BC) = C^TB^T(BC) = C^T(B^TB)C = C^T(I)C = C^TC = I$$

as desired.

This proof works in the exact same manner when the flip transpose is applied, meaning that we will be able to use this fact in Section 7 to look at the Jacobians created from the equations that arise from the matrix equation $A^FA = I$. Using this proposition, we can find the Jacobian at the identity matrix for our $2 \times 2 \in O_n(\mathbb{R})$,

$$dF_I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We can see that there is a pivot in each row, meaning the matrix is surjective. After seeing how to work with the equations for the 2×2 matrix case, we can move on to proving this fact for any size matrix A.

Theorem 10. *The orthogonal group is a submanifold.*

Proof. All polynomial equations created from the matrix equation $A^TA = I$ will have one of the two following forms

$$\sum_{k=1}^{n} a_{kr}^2 - 1 = 0 \ (1 \leq r \leq n) \qquad \sum_{k=1}^{n} a_{kr}a_{ks} = 0 \ (1 \leq r \leq s \leq n).$$

Remember that F is the map we are working with. This map has a matrix as its input and a vector as its output. We want to find a matrix that encompasses all of the possible expressions that will arise from $A^TA = I$, which will be referred to as the general form of the function. The general form of the function for any size matrix A is as follows

$$F(A) = \begin{bmatrix} \sum_{k=1}^{n} a_{k1}^2 - 1 \\ \vdots \\ \sum_{k=1}^{n} a_{kn}^2 - 1 \\ \sum_{k=1}^{n} a_{k1}a_{k2} \\ \vdots \\ \sum_{k=1}^{n} a_{k1}a_{kn} \\ \vdots \\ \sum_{k=1}^{n} a_{k(n-1)}a_{kn} \end{bmatrix}.$$

Notice that the orthogonal group is the set of solutions to

$$F(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

All of the equations that arise from the matrix equation are equal to 0. In other words, the inverse image of the zero vector gives all the matrices in $O_n(\mathbb{R})$. This is important because it satisfies the hypothesis of the Identity Check Trick. This will work similarly for $F_n(\mathbb{R})$.

To show that dF_A will be surjective for any matrix $A \in O_n(\mathbb{R})$, we will again use the Identity Check Trick. In other words, it is sufficient to check the case when $A = I$ to create the Jacobian

$$dF_I = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & & \cdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{bmatrix}.$$

The rank of the matrix is $\binom{n+1}{2}$, which is the number of expressions that were used to create each row. Therefore, dF_I is surjective. Since $O_n(\mathbb{R})$ is a subgroup and a submanifold of $GL_n(\mathbb{R})$, we can say that it is a Lie subgroup and hence a Lie group.

7 Flip Transpose Groups are Lie Groups

To show that $F_n(\mathbb{R})$ is a Lie group, we work in a similar manner to the proof of $O_n(\mathbb{R})$. Before finding the general form of the function, $F(A)$, and the Jacobian, dF_I, we need to work with some different sized matrices to find the pattern of equations that arise from $A^F A = I$. Let's begin with a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{so } A^F = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Setting $A^F A = I$, we get $\binom{2+1}{2} = 3$ expressions

$$F(A) = \begin{bmatrix} ad + bc - 1 \\ 2ac \\ 2bd \end{bmatrix}.$$

Using the Identity Check Trick, we can create the Jacobian

$$dF_I = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

We notice that this matrix has a pivot in each row and is therefore surjective. To help us gain an understanding of the pattern in the matrix, we repeat this process but use the indices of each entry in the matrix. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{then } A^F = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

and our three equations become

$$a_{22}a_{11} + a_{12}a_{21} = 1, 2a_{11}a_{21} = 0, \text{ and } 2a_{22}a_{12} = 0.$$

In general the equations take on one of the following three forms

$$\sum_{k=1}^{n} a_{n+1-k,1}a_{k,1} = 0 \quad \sum_{k=1}^{n} a_{n+1-k,2}a_{k,2} = 0 \quad \sum_{k=1}^{n} a_{n+1-k,2}a_{k,1} - 1 = 0.$$

Notice that the index $n+1-k$ is arising again in our study of the flip transpose group. This index originally arose when we were proving properties of $F_n(\mathbb{R})$ in Section 4.

Let's now look at the 3×3 case when

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ and } A^F = \begin{bmatrix} i & f & c \\ h & e & b \\ g & d & a \end{bmatrix}.$$

Setting $A^F A = I$, we get $\binom{3+1}{2} = 6$ expressions

$$F(A) = \begin{bmatrix} 2ag + d^2 \\ 2bh + e^2 - 1 \\ 2ic + f^2 \\ ha + ed + bg \\ ib + fe + ch \\ ia + fd + cg - 1 \end{bmatrix}.$$

Using the Identity Check Trick, we create the Jacobian

$$dF_I = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We see again that the Jacobian has a pivot in every row and is therefore surjective. We also see that the equations that arise from the matrix equation $A^F A = I$ all equal 0, meaning that if we take the inverse map, F^{-1}, we will get all matrices in $F_n(\mathbb{R})$. After using indexed entries again, we can generalize the equations into the following forms

$$\sum_{k=1}^{n} a_{n+1-k,1}a_{k,1} = 0 \sum_{k=1}^{n} a_{n+1-k,2}a_{k,2} - 1 = 0 \sum_{k=1}^{n} a_{n+1-k,3}a_{k,3} = 0$$

$$\sum_{k=1}^{n} a_{n+1-k,2}a_{k,1} = 0 \sum_{k=1}^{n} a_{n+1-k,3}a_{k,1} - 1 = 0 \sum_{k=1}^{n} a_{n+1-k,3}a_{k,2} = 0. \tag{1}$$

We begin to see that when the second components of each of the terms in the summation sum to $n+1$, the equations in (1) are subtracting one. This observation will be important when creating the general form of the function.

To arrive at a general form of the function for the polynomial equations created from the matrix equation $A^F A = I$, we repeat these steps for a 4×4 matrix hoping to get summations similar to the ones that came from working with the indexed entries of the 2×2 and 3×3 matrix cases. We first find the Jacobian using the Identity Check Trick. The Jacobian yields the same pattern as those that we found before

$$dF_I = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We notice that there are n 2's in the top 4 rows of the matrix and they are spaced 2 columns apart. This matrix is also surjective, meaning the 4×4 matrices in $F_n(\mathbb{R})$ form a Lie group. When we use indices for the 4×4, we begin to see a pattern in the summations that can be created from the polynomials that are arising. After repeating these steps for the 5×5, we find the same pattern. We can see that the following matrix expresses the polynomials that arise from $A^F A = I$ for any matrix A,

$$F(A) = \begin{bmatrix} \sum_{k=1}^{n} a_{n+1-k,1} a_{k,1} \\ \sum_{k=1}^{n} a_{n+1-k,2} a_{k,2} \\ \vdots \\ \sum_{k=1}^{n} a_{n+1-k,n} a_{k,n} \\ \sum_{k=1}^{n} a_{n+1-k,2} a_{k,1} \\ \vdots \\ \sum_{k=1}^{n} a_{n+1-k,i} a_{k,j} \\ \vdots \\ \sum_{k=1}^{n} a_{n+1-k,n} a_{k,n-1} \end{bmatrix}$$

where $i>j$ and $i+j=n+1$ when the expression is subtracting 1.

The Jacobian matrix of F at $A = I$ is the $\binom{n+1}{2} \times n^2$ matrix dF_I. To show that dF_I is surjective for any $A \in F_n(\mathbb{R})$, we will use the following argument.

One way for the Jacobian to always be surjective is that there needs to be exactly one non-zero entry in every column and at least one non-zero entry in every row. The Jacobians created for the $2 \times 2, 3 \times 3,$ and 4×4 all exhibited these two ideas, hence there was a pivot in every row and the matrix was surjective. We have noticed that in the top block of each Jacobian there are n 2's. For the top block of any Jacobian, in any one equation, one unique diagonal entry (a_{ii}) shows up twice (if n is even) or is squared (if n is odd). Hence when we take the derivatives at the identity, we get 2's. For the bottom block of the Jacobian, which has a random pattern of two 1's in every row, in any one equation, there will be two diagonal entries that show up only once. To back up this argument, we will use the above matrix to model various patterns of equations that will arise in any size matrix A. If we look at the first equation that will always arise,

$$\sum_{k=1}^{n} a_{n+1-k,1}a_{k,1}$$

and we make $k = 1$, we get $a_{n1}a_{11}$, so we have a diagonal entry, a_{11}. For our argument to stand, this entry needs to show up only one more time in this equation, and that happens when $k = n$, because we get a_{11}, a_{n1}.There are no other values of k that will cause this diagonal entry to arise in the equation. This is where our 2 would come from. If we move to the bottom block, where the first equation is of the form

$$\sum_{k=1}^{n} a_{n+1-k,2}a_{k,1}$$

we need to show that in any one equation exactly 2 diagonal entries show up only once. Suppose $k = 1$, then we have $a_{n2}a_{11}$. If we let $k = n - 1$, we have $a_{22}a_{n-1,1}$. We see that we get 2 different diagonal entries, a_{11} and a_{22} that will not show up again in that equation. Upon repeating this process for various equations with various values of k, the argument stands. Therefore, there will be at least one non-zero entry in every row. For every column to have exactly one non-zero entry, each variable is only ever once multiplied by the a_{ii} entry.

Since $F_n(\mathbb{R}) \leq GL_n(\mathbb{R})$ and $F_n(\mathbb{R})$ is a submanifold, it is a Lie subgroup. We can conclude that the flip transpose group is a Lie group.

8 Further Considerations

The work that has been done in this paper shows that the flip transpose group is a Lie group. As far as we know, this matrix group has not been looked at in depth before. Since $F_n(\mathbb{R})$ is a Lie group, but not immediately recognizable as a standard Lie group, possible further work with $F_n(\mathbb{R})$ could be aimed at investigating which classification this new matrix group would fall under.

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